

MATHEMATICAL MODELS FOR SOCIAL AND ECONOMIC DYNAMICS AND FOR TAX EVASION: A SUMMARY OF RECENT RESULTS

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1 Introduction

This paper presents a collection of results on the modelling of social phenomena, such as economic dynamics, tax evasion, diffusion of criminality. A large part of the more recent results that we will describe were obtained by our research group (Iacopo Borsi, Giorgio Busoni, Antonio Fasano, Alberto Mancini, Luca Meacci, Mario Primicerio) with the collaboration of Miguel A. Herrero and Juan Carlos Nuno of the Univeristy Complutense in Madrid, and involving some of our students as well. See [8, 9, 10, 19, 20, 21, 22].

2 Economic dynamics: compartmental models

The simplest mathematical framework that we can use to describe the evolution of the distribution of wealth in a society is based on a compartmental model. We consider a closed population composed of N individuals and we identify n sub-populations (classes) U_1, U_2, \dots, U_n of increasing wealth. If we denote by $u_i(t)$ the number of individuals that, at time t , are in the class U_i we have

$$u_1(t) + u_2(t) + \dots + u_n(t) = N \quad (1)$$

Next, we assume that the rate of transition from any class k to the adjacent classes is linear and governed by suitable non-negative coefficients of "social promotion" α_k

and of “social relegation” β_k , while the transition to a social class not adjacent is forbidden.¹

Therefore, we will have the following system of ordinary differential equations:

$$\dot{u}_k(t) = \alpha_{k-1}u_k(t) - (\alpha_k + \beta_k)u_k(t) + \beta_{k+1}u_{k+1}(t), \quad k = 1, 2, \dots, n \quad (2)$$

where $\alpha_0 = \beta_{n+1} = \beta_1 = \alpha_n = 0$. It is obvious that if α_k and β_k are constant for any k , the stationary solution² has to satisfy

$$\hat{u}_k = \frac{\alpha_1}{\beta_2} \frac{\alpha_2}{\beta_3} \dots \frac{\alpha_{k-1}}{\beta_k} \hat{u}_1 \equiv \gamma_{k-1} \hat{u}_1, \quad k = 2, \dots, n \quad (3)$$

and hence is uniquely determined if we impose the condition (1),

$$\hat{u}_1 = \frac{N}{(1 + \sum_{k=2}^n \gamma_{k-1})}, \quad (4)$$

Now, if we define $\gamma_0 = 1$ we can write

$$\hat{u}_k = \frac{N\gamma_{k-1}}{\sum_{j=1}^n \gamma_{k-1}} \equiv \omega_k N \quad (5)$$

In practice, mobility coefficients are not constant but depend on the social dynamics itself. For instance in [16] the case is considered when they depend on the dimension N of the population, assuming that each class has a specific demographic behaviour. But even in the case of isolated population an interesting situation arises when α_k and β_k are assumed to depend on the total wealth of the population. The latter can be expressed as

$$W(t) = \sum_{k=1}^n p_k u_k(t), \quad (0 \leq p_1 \leq p_2 \leq \dots \leq p_n) \quad (6)$$

where p_k is the contribution that each individual in U_k gives to the common richness.

A more sophisticated model, instead of (6) could take into account the budgetary policy of the government in the form of another ordinary differential equation

$$\dot{W}(t) = \sum_{k=1}^n p_k u_k(t) - \Psi(W(t), t), \quad (7)$$

¹Of course, this assumption could be released and coefficients α_{ki} ($k < i$) and β_{ki} ($k > i$) could be introduced. We chose the approach above to give a simpler and more evident picture.

²Note that (1) excludes the trivial solution of (2), since $N > 0$.

where, in this case, p_k is the amount of taxes paid by the individual in class U_k and Ψ is the rate of expenditure of the community.

To characterize the possible stationary solutions of (2), (6), one can use (5) and say that the system admits as many stationary solutions as the fixed points of the transformation

$$W = N \sum_{k=1}^n p_k \omega_k(W). \quad (8)$$

To be specific, consider the toy model with $n = 2$

$$\dot{u}_1 = \beta u_2 - \alpha u_1, \quad \dot{u}_2 = \alpha u_1 - \beta u_2, \quad W = p_1 u_1 + p_2 u_2. \quad (9)$$

Normalizing and setting

$$x = u_1/N, \quad y = u_2/N, \quad R = W/p_2 N, \quad (10)$$

we have that the stationary wealth \hat{R} has to satisfy

$$\hat{R} = \frac{\alpha + \rho\beta}{\alpha + \beta}, \quad \text{with } \rho = p_1/p_2, \quad 0 < \rho < 1. \quad (11)$$

The r.h.s. is a given function of \hat{R} so that (11) can be written as

$$\hat{R} = \phi(\hat{R}), \quad (12)$$

where ϕ is continuous and $0 < \phi < 1$, so that (12) has at least one solution in $(0, 1)$. It turns out that this is unique if $\phi' < 1$. It can be easily checked that it is globally asymptotically stable.

3 Economic dynamics: continuous models

3.1 The balance equation

A generalization of the model described in the previous section consists in assuming that each individual at a given time t has a wealth index x (where x is any real non-negative number in a given interval that can represent the salary, the amount of income tax, etc.: we will normalize x and assume that it can take any values in $[0, 1]$). We will describe the wealth distribution in the society by an L^1 function $n(x, t)$ such that the integral

$$\int_{x_1}^{x_2} n(x, t) dx, \quad 0 \leq x_1 < x_2 \leq 1,$$

represents the number of individuals that, at time t , have wealth index between x_1 and x_2 .

The mobility in the society is represented by a function $\gamma(x, y)$ such that

$$\int_{t_1}^{t_2} dt \int_0^1 dy \int_{x_1}^{x_2} n(x, t) \gamma(x, y) dx,$$

represents the number of individuals that, in the time interval (t_1, t_2) , leave the class with wealth index belonging to the interval (x_1, x_2) . Similarly, a number

$$\int_{t_1}^{t_2} dt \int_0^1 dx \int_{x_1}^{x_2} n(x, t) \gamma(x, y) dy,$$

enters the class (x_1, x_2) in the time interval (t_1, t_2) .

The balance equation takes the form

$$\frac{\partial n(x, t)}{\partial t} = -n(x, t) \int_0^1 \gamma(x, y) dy + \int_0^1 n(y, t) \gamma(y, x) dy, \quad (13)$$

that has to be solved with a prescribed initial condition

$$n(x, 0) = n_0(x). \quad (14)$$

It can be proved [9] that if $n_0 \in L^1(0, 1)$, $n_0 \geq 0$ and $\gamma \in L^\infty(0, 1)^2$, $\gamma \geq 0$, then (13), (14) has a unique solution that is non-negative, belongs to $L^1(0, 1)$ for any positive t and is continuously differentiable w.r.t. t . Moreover,

$$\int_0^1 n(x, t) dx = \int_0^1 n_0(x) dx = N. \quad (15)$$

Among the regularity properties that can be proved (see [10]), we quote the fact that the differentiability of n_0 and γ yields the same property (w.r.t. x) for $n(x, t)$.

As a simple example, consider the case

$$\gamma(x, y) = q(y). \quad (16)$$

If

$$a \equiv \int_0^1 q(y) dy > 0, \quad (17)$$

we can find the explicit form of the solution of (13)-(14)

$$n(x, t) = \frac{N}{a} q(x) [1 - e^{-at}] + n_0(x) e^{-at}. \quad (18)$$

We note that more general models can be studied with little additional work. For instance γ may be allowed to depend on t as well, or even on the total wealth that, in the present case, is given by

$$W(t) = \int_0^1 p(x)n(x, t)dx \tag{19}$$

with $p(x)$ monotonically increasing.

The dependence on age a (which could be crucial when economic dynamics is coupled with criminology, see [12]) could also be included in the balance equation. But in that case additional complication is introduced for the presence of boundary condition on $a = 0$, in particular when this condition is given in terms of the fertility of the population.

3.2 Stationary solutions

For the sake of simplicity we will deal with the special case in which $n_0(x) \in C[0, 1]$ and $\gamma(x, y) \in C([0, 1]^2)$, and we will exclude the trivial case $\gamma \equiv 0$.

Define

$$g(x) = \int_0^1 \gamma(x, y)dy, \quad h(x) = \int_0^1 \gamma(y, x)dy. \tag{20}$$

We can prove ([9]) that, if $g(x) > 0$ in $[0, 1]$, then (13) admits a constant stationary solution

$$n_\infty(x) = N, \quad x \in [0, 1], \tag{21}$$

if and only if

$$g(x) = h(x), \quad x \in [0, 1]. \tag{22}$$

As an example, consider the particular case

$$\gamma(x, y) = p(x)q(y), \quad p(x) > 0, \quad q(y) \geq 0. \tag{23}$$

Then (13) has the stationary solution

$$n_\infty(x) = N \frac{q(x)}{p(x)} \left(\int_0^1 \frac{q(z)}{p(z)} dz \right)^{-1}. \tag{24}$$

In particular the stationary solution is constant if and only if

$$q(x) = Ap(x), \tag{25}$$

for any positive constant A .

3.3 Inequality index

It is well known that, since the classical paper [18], several indexes have been proposed to characterize the economic inequality in a society, see the celebrated Gini's paper [13] and the references in [17]. We propose here an index that is simply related to the average wealth. Set in (19)

$$p(x) = x, \quad (26)$$

(a choice which is reasonable and in any case does not imply a loss of generality since $p(x)$ is monotonically increasing with x) and define the average per capita wealth as

$$\hat{W}(t) = N^{-1} \int_0^1 xn(x, t)dx. \quad (27)$$

We define the inequality index

$$i(t) = \frac{1}{N\hat{W}(t)(1-\hat{W}(t))} \int_0^1 (x - \hat{W}(t))^2 n(x, t)dx, \quad (28)$$

whence we have immediately

$$0 \leq i \leq 1. \quad (29)$$

We note that for a population in which all the members have the same wealth (necessarily \hat{W} ; from now on we do not indicate time dependence explicitly, to simplify notation) we would have $n(x) = N\delta(x - \hat{W})$, where δ is the Dirac distribution. In this case $i = 0$.

On the contrary, for the same total wealth $N\hat{W}$ the most unequal distribution corresponds to the case in which $N\hat{W}$ individuals have wealth index 1 and $N(1 - \hat{W})$ have wealth index 0. This corresponds to

$$n(x) = 2(1 - \hat{W})\delta(x) + 2N\hat{W}\delta(x - 1), \quad (30)$$

(the factor 2 is introduced since $\int_0^1 \delta(x)dx = \int_0^1 \delta(x - 1)dx = 1/2$). Hence, in this case,

$$i = \frac{1}{\hat{W}(1 - \hat{W})} \left[(1 - \hat{W})\hat{W}^2 + \hat{W}(1 - \hat{W})^2 \right] = 1. \quad (31)$$

To see, on some other examples, how index i depends on the distribution function $n(x)$, we consider the case

$$n(x) = N\alpha x^a(1 - x)^b, \quad (32)$$

where $a, b \in \mathbb{N}$ and where α is a normalization constant such that $\int_0^1 n(x)dx = N$. After some lengthy algebra we get

$$i = \frac{1}{a + b + 3}. \quad (33)$$

3.4 Some numerical simulations

We give here some mathematical example taken from paper [9].
Let us take the mobility function

$$\gamma(x, y) = \beta(x)\theta(|x - y|)H(x - y) + \alpha(x)\theta(|x - y|)H(y - x) \quad (34)$$

where

- H is the Heaviside function,
- $\alpha(x) = (1 - x)$,
- $\beta(x) = x$,
- $\theta(z) = e^{-z^2/2\sigma^2}$ (with $\sigma = 0.3$, $z = |x - y|$) is a function modulating the kernel with the distance between x and y .

Let us take four different initial conditions (with the same initial population $N = 1/6$)

$$n_1(x) = \frac{1}{3s\sqrt{2\pi}}e^{-x^2/2s^2}, \quad s = 0.2, \quad (35)$$

$$n_2(x) = \frac{1}{6s\sqrt{2\pi}}e^{-(x-0.5)^2/2s^2}, \quad s = 0.2, \quad (36)$$

$$n_3(x) = \begin{cases} -a(x - x_0)/x_0, & 0 < x \leq x_0, \\ b(x - (1 - x_0))/x_0, & (1 - x_0) \leq x \leq 1, \\ 0, & \text{elsewhere,} \end{cases} \quad (37)$$

with $x_0 = 1/8$, $a = 2$, $b = (1/3 - ax_0)/x_0$,

$$n_4(x) = \frac{1}{6}(1 + \sin(4\pi x)), \quad (38)$$

and compute $n(x, t)$. We use a finite difference scheme, with an explicit forward method in time. The integrals appearing in the equations are solved by the trapezoidal rule integration method. The computation shows that the equilibrium solution corresponding to the four initial conditions coincide (Fig. 6). On the other hand, the evolution of $i(t)$ is obviously different (Fig. 2).

As we did in [9], we now consider an inverse problem for the integro-differential equation (13) i.e. we look for a function $\tilde{\gamma}(x, y)$ such that (13) admits a given stationary solution $\tilde{n}(x)$. Of course, we do not expect that such problem is uniquely solvable (in any case $\tilde{\gamma}$ is defined up to a multiplicative constant). This is clearly shown noting that given $\tilde{n}(x) \geq 0$, for any

$$\tilde{\gamma}(x, y) = p(x)p(y)\tilde{n}(y), \quad (39)$$

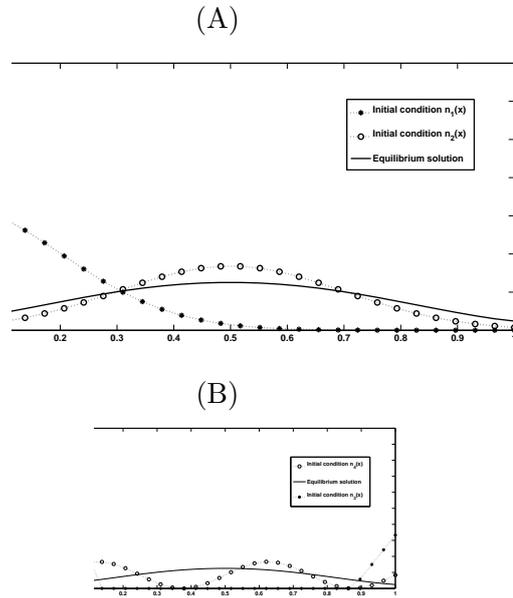


Figure 1: *Reaching the equilibrium solution from different initial data. The equilibrium solution (solid line) is the same in all cases, which have been split in two figures only for the reader convenience. (A): Initial conditions (35) and (36). (B): Initial conditions (37) and (38).*

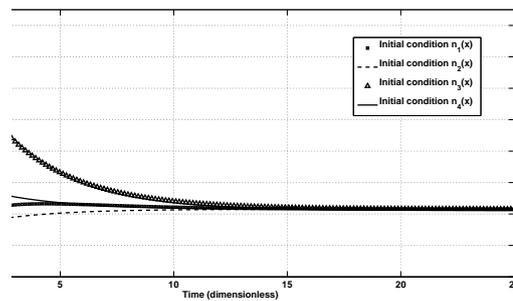


Figure 2: *Inequality index computed using the mobility function (34) and initial conditions (35), (36), (37) and (38).*

where $p(x)$ is an arbitrary positive function, equation (13) has the time-independent solution $\tilde{n}(x)$.

For a numerical check, let us take $n_3(x)$ (see (37)) to play the role of target equilibrium solution and prove that taking

$$\tilde{\gamma}(x, y) = p(x)p(y)n_3(y) \tag{40}$$

for any positive $p(x)$, problem (P) has the asymptotic solution $n_3(x)$, for any initial datum. Fig. 7 and 8 verify this fact, taking $n(x, 0) = n_4(x)$ as initial condition and taking two different functions $p(x)$.

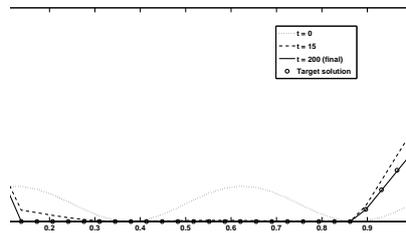


Figure 3: An example of result stated in (39), with $n(x, 0) = n_4(x)$ and $p(x) = x + 0.1$.

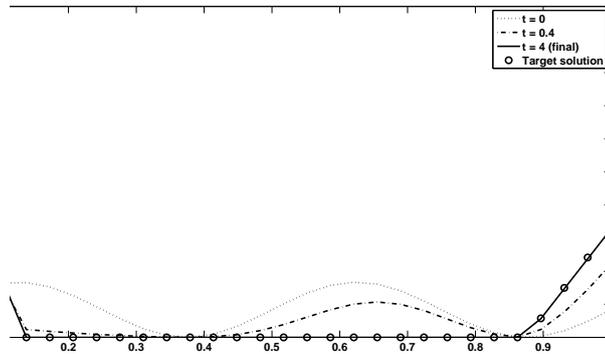


Figure 4: An example of result stated in (39), with $n(x, 0) = n_4(x)$ and $p(x) = 1.5 + \sin(2\pi x)$.

3.5 A PDE approach

Consider the second integral in (13) and expand $n(x, t)$ assuming that n is a smooth function:

$$n(y, t) = n(x, t) + n_x(y, t)(y - x) + n_{xx}(y, t)\frac{(y - x)^2}{2} + \dots \quad (41)$$

Since it is reasonable to think that γ has support in a sufficiently “small” strip centered around the diagonal of the square $(0, 1) \times (0, 1)$, we approximate (13) by

$$\frac{\partial n(x, t)}{\partial t} = C(x)n(x, t) + B(x)n_x(x, t) + D(x, t)n_{xx}(x, t), \quad (42)$$

with

$$C(x) = \int_0^1 [\gamma(y, x) - \gamma(x, y)] dy, \quad (43)$$

$$B(x) = \int_0^1 (y - x)\gamma(y, x) dy, \quad (44)$$

$$D(x) = \int_0^1 \frac{(y - x)^2}{2}\gamma(y, x) dy. \quad (45)$$

Because of our assumptions on the support of γ the integrals in (43), (44), (45) may be computed over the interval $([x - \delta]_+, \max(1, x + \delta))$. It is immediately seen that (42) can be written in a divergence form

$$n_t = (D(x)n_x + (B(x) - D'(x))n)_x + (D''(x) - B'(x) + C(x))n. \quad (46)$$

Here we provide a simple numerical example, assuming

$$\gamma(x, y) = \begin{cases} 0.3 \exp\left[-\left(\frac{y - 0.5}{2\delta^2}\right)^2\right], & \text{if } |x - y| < \delta, \\ 0, & \text{otherwise.} \end{cases} \quad (47)$$

with $\delta = 0.1$. Then (46) is a uniformly parabolic equation and if we impose as boundary condition

$$D(x)n_x + (B(x) - D'(x))n = 0, \text{ on } x = 0 \text{ and } x = 1,$$

then the quantity $\int_0^1 n(x, t) dx$ is conserved up to $\mathcal{O}(\delta)$ for any fixed time t . Assuming the following initial condition

$$n_0(x) = 0.7 \exp\left[-\left(\frac{x - 0.4}{\sigma}\right)^2\right] + 0.3 \exp\left[\left(\frac{x - 0.6}{\sigma}\right)^2\right] \quad (48)$$

with $\sigma = 0.03$, we solved the integro-differential equation (13) and the partial differential equation (42), both for $t \in [0, T]$, with $T = 100$. In Fig. 5 a comparison of the two solutions evaluated at final time is shown. The initial condition is also displayed, in order to show the redistribution effect obtained by imposing a transition function $\gamma(x, y)$ centered in the middle wealth index 0.5. It is worthy to note that the global absolute difference between solutions (evaluated for any $x \in [0, 1]$, at any time) was computed as $E = 0.1183$ which is in agreement with the error estimate in the Taylor approximation (41), namely $E \approx \mathcal{O}(T\delta^3) \approx 0.1$

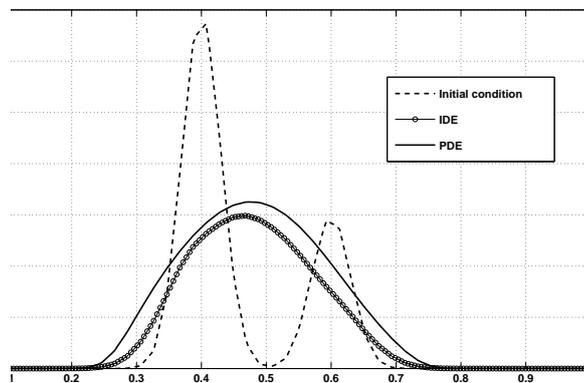


Figure 5: Comparison between solution of (13), integro-differential equation (IDE) and (42), partial differential equation (PDE), at final time $T = 100$. The initial condition (48) is also displayed..

4 Mathematical criminology

The use of mathematical models in criminology does not have a large and rich tradition. Nevertheless, these topics attract a growing attention (see e.g. the special volume of EJAM [3] where a great deal of references can be found [14] and some recent workshops [1, 2]).

Essentially, we can generally say that a problem in criminology consists of

- (a) An observable: the number of crimes (of a specific type) as a function of time and position.
- (b) The state variables that one wants to take into account, e.g.
 - age and income distribution
 - mobility
 - school, housing, social and urban segregation
 - topology of targets
 - crime patterns, crime organization
- (c) The control functions as for instance
 - police forces and strategy
 - social control
 - law enforcement (severity of punishment)
 - social policy, welfare, school policy.

Although mathematical models might be unable to make quantitative predictions (e.g. number of burglaries that will happen in a given urban domain over next month), they can suggest the qualitative behaviour of a given social system and simulate how the outcomes change when control functions are changed.

Thus, they can be used to plan strategies to contrast criminology and to employ suitable resources in an optimal way.

Moreover, they are instrumental in the use and interpretation of the enormous amount of data that are stored in the archives.

Roughly speaking, one could identify three main branches of mathematical criminology, according to the class of methods and models used:

- (i) models based on game theory
- (ii) models based on population dynamics

(iii) agent-based models.

We shall mostly concentrate on population dynamics and will finally deal with a special class of models that can be included in the third category, i.e. models based on cellular automata.

4.1 Criminology and population dynamics: “ecological” models

This class of models, in its different variants, is basically a predator-prey system. In the simplest case, one has three sub-populations: the targets T , the criminals C , and the guards G . Of course, T are preys for C , that in turn are “predated” by G . In some cases G could also be considered predators w.r.t. T since the latter sub-population bears the costs of maintaining the guards.

The main mathematical information that can be drawn from these models concerns the study of the equilibrium points and their character (stability, attractivity and relevant domain, etc.)

A specific case was considered in a paper by Vargo [24]. Here, just C and G were considered and in the system

$$\begin{cases} \dot{C} = aC - bCG + A \\ \dot{G} = -\alpha C + \beta CG + B \end{cases} \quad (49)$$

A and B represent the influence of external “world” on the two populations.

A more complex model with three sub-populations is considered in [20]. Basically, the system is

$$\begin{cases} \dot{T} = r(N - T)(k - T) - \alpha TC - \beta G \\ \dot{C} = f\alpha TC - \gamma CG - \mu C \\ \dot{G} = g\alpha TC - h\gamma CG - mG \end{cases} \quad (50)$$

In (50) the number of crimes (proportional to the product TC) determines the increase of the number of guards, but also the recruitment of new criminals.

The system (50) is particularly rich, in particular when the terms accounting for the effect of predation are substituted by functions of Hollig’s type. But also when T is assumed to be constant interesting bifurcation behaviours are observed as a function of parameter γ representing the efficiency of security forces.

The paper [4] considers the interactions between the population of drug smugglers b and drug producers n . The interaction between the two sub-populations is symbiotic and the corresponding dynamical system is

$$\begin{cases} \dot{b} = -h_1 b + k_1(\bar{B} - b)n \\ \dot{n} = -h_2 n + k_2(\bar{N} - n)b \end{cases} \quad (51)$$

where \bar{B} and \bar{N} are the total number of potential smugglers and producers, respectively.

4.2 Criminology and population dynamics: “epidemiological models”

In this class of models, the recruitment in the sub-population of criminals is supposed to occur by a sort of “contagion” of a part of the population by the criminals.

For instance, in the paper [23] the following system is considered

$$\begin{cases} \dot{N} = -\theta N + \mu S + (A + BN)C + \gamma P + \Pi\phi_2 P \\ \dot{S} = \theta N - \mu S - \alpha S - \lambda SC + -\gamma P \\ \dot{C} = \alpha S + \lambda SC - (A + BN)C - \phi_1 C + (1 - \Pi)\phi_2 P \\ \dot{P} = \phi_1 C - \phi_2 P \end{cases} \quad (52)$$

Here the class P of prisoners is introduced and the non-criminals are divided in two sub-classes: S are susceptible to the contagion and may be recruited by criminals C , while N are non-susceptible. No “predation” terms are considered.

It can be noted that a “contagion” is also acting between N and C : the presence of N induces a fraction of C to “redeem”. Among the most questionable weaknesses of the model we can quote:

- (i) the fact that no input is considered from C to S nor from P to S .
- (ii) the fact that “deterrence” on S from becoming criminals is determined by the number of prisoners. As a consequence, we remark that (52) formally does not preserve the positivity of the solutions.

But it can be noticed that the report quoted above contains a careful calibration of the parameters based on well-structured historical data. The analysis of equilibrium points allows the authors to simulate different scenarios corresponding to different policies, i.e. to different set up of parameters. We can also note that the population of guards is not specifically considered, but that it could be included by mixing the two types of models.

It is clear that models of this class can also take some “social” factors into account. For instance, in the scheme considered in [26] a population of (temporarily) “recovered” is introduced whose inputs come from the susceptible class (or the “poor” in the spirit of the paper) and from released from jail. The

system is:

$$\begin{cases} \dot{N} = -\mu T - (\sigma + \mu)N \\ \dot{S} = \sigma N - \beta SC - \gamma S - \mu S \\ \dot{C} = \beta SC + \phi BRC - \rho C - \mu C \\ \dot{P} = \rho C - \delta P - \mu P \\ \dot{R} = \gamma S + \delta P - \phi BRC - \mu R \end{cases} \quad (53)$$

In (53) $\phi < 1$ is the reduction of the efficiency of contagion due to experience of jail, T is the total dimension of the population. The latter is assumed to be constant so that if μ is the death rate of each sub-population, the aggregate birth rate is also μ (and all individuals are supposed to born in the class N , a fact that is rather questionable).

4.3 Criminology and mathematical models: a more complex scheme

A general model taking into account the socio-economic dynamics discussed in Sec. 2 and the dynamics of recruitment, arrest, etc. of criminals will be composed of $n + 3$ populations, where n is the number of classes. Besides of coefficients α_j and β_j of social promotion and relegation of each social classes, one has to prescribe the following rates:

- recruitment rate of criminals R_j (in principle from each social class);
- crime rate K_j (in principle affecting each class);
- arrest rate A ;
- “spontaneous” decay D of criminals;
- release rate F_j of prisoners (in principle entering each class);
- hiring rate H_j of guards;
- “induced” decay of guards L ;
- “spontaneous” decay of guards D .

Just to simplify a bit we assume that criminals are recruited in the poorest class and write³

³The bilinear terms $\theta_j u_j(t)C(t)$ and $mC(t)G(t)$ can be substituted by terms in the Hollig’s form, $\theta_j u_j C / (l_i + u_i)$ and $mCG / (l + C)$.

$$\begin{aligned}
R &= ku_1(t)C(t) \\
K_j &= \theta_j u_j(t)C(t), \quad K(t) = \sum_j K_j \\
A &= mC(t)G(t) \\
D &= -\rho C(t) - \nu C^2(t), \text{ taking into account some intra-specific competition} \\
F &= -\tau P(t) \\
H &= \sum_j H_j = hK(t) \\
L &= -\delta A \\
D &= -qG(t).
\end{aligned}
\tag{54}$$

If one thinks that some of the parameters listed above may depend on the actual wealth of the society (that is in turn depending on the $u_j(t)$), it is clearly understood that the task of getting quantitative information from the model in real situations is hopeless, and maybe meaningless because of the huge number of parameters to be calibrated.

In any case, from the theoretical point of view, the well-posedness of the complete model can be proved, as well as the positivity of all the unknown functions.

In the sequel we present a concrete example that shows that some qualitative information can be obtained.

Assume $n = 2$ (two social classes: the poor and the rich individuals). Moreover we take the police size as a given constant and we disregard the population in jail. This means that our analysis is done on an intermediate scale between the average time in jail and the time required to change the policy of hiring police forces. Thus the model is the following

$$\begin{cases} \dot{u}_1 = -\alpha u_1 + \beta u_2 - ku_1 C + \rho C + \nu C^2 + mCG/(l + C) \\ \dot{u}_2 = \alpha u_1 - \beta u_2 \\ \dot{C} = ku_1 C - \rho C + \nu C^2 - mCG/(l + C) \end{cases}
\tag{55}$$

We also assume that k depends on the total wealth $W(t)$ whose dynamics is expressed by

$$\dot{W}(t) = a_1 u_1 + a_2 u_2 - \theta W - \lambda(W) \frac{u_2 C}{l_2 + u_2} - g(W)G.
\tag{56}$$

Here θ accounts for the budget policy of the community, λ measures the negative effect on the wealth due to the crimes (committed only against the rich people) and G is the (given) number of guards while g is the rate of expenditure for their maintenance. To be specific, we suppose that both λ and g depend linearly on W and discuss the equilibria and their stability in terms of k and

G . Consider in particular the criminal-free equilibrium $C = 0$. The stability of this equilibrium is ensured if and only if

$$ak - b - cG - dG^2 < 0, \tag{57}$$

where a, b, c, d are positive numbers that can be calculated from the parameters of the system. Thus, if k exceeds a critical value depending quadratically on G , a new equilibrium state exists with non-vanishing population of criminals.

The model can also be used to discuss the budgetary policy of the society. For this reason we disaggregate in (56) the expressions that are devoted to social promotion, writing

$$\theta = s + \hat{\theta}, \tag{58}$$

so that sW is the rate of “social” expenditures whose effect is modelled assuming that α (i.e. the coefficient of social promotion) is increasing with sW .

Summing up $(s + gG)W$ is the rate of the expenditures devoted to fight criminality (“how much”) whereas the factor s or the ratio gG/s indicates “how” the society decides to split this expenditure between repression of the crime and social promotion.

Using the model above, it is possible to see if this choice can be optimized (maximization of wealth, minimization of criminality, etc.) over a given time horizon. Of course, an appropriate cost function has to be defined and minimized.

5 A model based on cellular automata

We consider a particular kind of crime i.e. tax evasion.

Let us study the abstract situation in which an individual (i.e. a cell in the 2D cellular automaton) can be either in the state X (tax-payer) or Y (tax-evader) and that the transition from one state to another is influenced by the state of its neighbours and by a global field.

5.1 No law enforcement

We assume ([19]) that the population occupies every cell of a square grid of side n , so that the total dimension of the population is $N = n \times n$. In our case, all the simulations have been performed with $n = 40$. We define the probability of changing the current state as sum of these two contributions (see [7], [6]):

$$P^{TOT} = P^{LOC} + P^{GLO} \tag{59}$$

where P^{LOC} and P^{GLO} are the local and global probabilities of changing the state. Obviously, the probability of remaining in the same state is given by $1 - P^{TOT}$.

Let us first define the local probabilities. The local probability that a taxpayer becomes a cheater in the next time step is given by:

$$P_{X \rightarrow Y}^{LOC} = l \frac{N_{LY}}{N_L}, \quad 0 < l < 1 \quad (60)$$

where N_{LY} is the number of cheaters in the neighborhood of this cell formed by N_L cells (so, $\frac{N_{LY}}{N_L}$ is the fraction of cheaters in its neighborhood). In the same way we can define the local probability that a cheater becomes a tax-payer as

$$P_{Y \rightarrow X}^{LOC} = k \frac{N_L - N_{LY}}{N_L}, \quad 0 < m < 1. \quad (61)$$

Obviously, $N_L - N_{LY}$ corresponds to the number of taxpayers in the neighborhood.

Next, we define the global probability for a tax payer to become a cheater as

$$P_{X \rightarrow Y}^{NONLOC} = \tau, \quad 0 < \tau < 1 \quad (62)$$

and, similarly, the global probability for a cheater to become a tax payer as

$$P_{Y \rightarrow X}^{NONLOC} = \alpha, \quad 0 < \alpha < 1. \quad (63)$$

We use a synchronous or parallel updating rule where all the system sites are updated at the same time step (in contrast to asynchronous or sequential updating where only one randomly selected site is update at each time) [11, 25]. We assume reflection boundary conditions on the sides of the square. The model is then completed by defining the neighborhood of each cell as a square centered in the cell and formed by $m \times m$ cells excluding the cell itself (the so-called Moore neighborhood [15]). In the sequel, we will specify the value of m .

In the spirit of classical population dynamics [5], we write an ODE expressing the balance in terms of inflow and outflow. Thus, we write:

$$\frac{dY}{dt} = \dot{Y} = \tau^* (N - Y) - \alpha^* Y + \frac{l^*}{N} Y (N - Y) - \frac{k^*}{N} (N - Y) Y \quad (64)$$

where the coefficients in the contagion terms are written as $\frac{l^*}{N}$ and $\frac{k^*}{N}$ for normalization purposes and $\tau^*, \alpha^*, l^*, k^* \geq 0$.

Normalizing and defining the new parameter:

$$d = l - k \quad (65)$$

we have to solve the following Initial Value Problem:

$$\begin{aligned} \dot{y}(t) &= \tau(1-y) - \alpha y + d y(1-y) \\ y(0) &= y_0 \end{aligned} \tag{66}$$

An explicit expression of the solution of the Initial Value Problem (66) for the case $d > 0$ can be found and its asymptotic value is given by the branch between 0 and 1 of the following expressions:

$$\bar{y} = \frac{d - (\tau + \alpha) \pm \sqrt{(\tau + \alpha)^2 + 2d(\tau - \alpha) + d^2}}{2d} \tag{67}$$

Clearly, if we assume that N_L is the entire square or, equivalently, that $m = n$, the CA is a probabilistic approach to the finite difference approximation of equation (66).

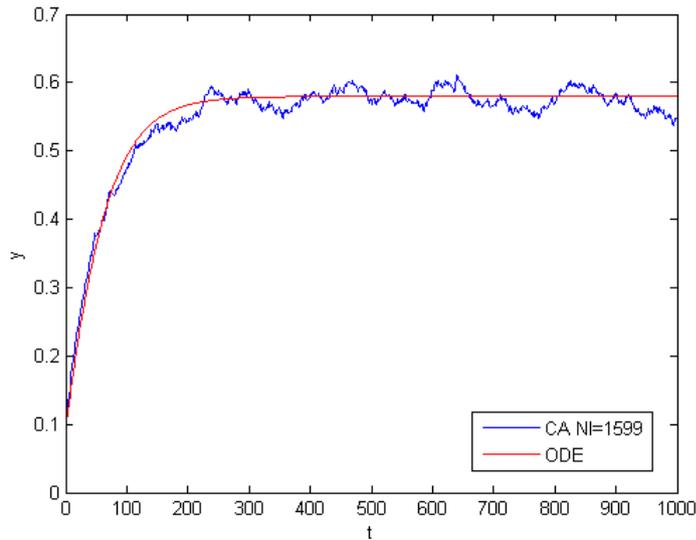


Figure 6: Time evolution of the cheater population y of the ODE and CA with maximum neighborhood ($N_L = 1599$). The parameter setup is $\alpha_0 = 0.01$, $\tau_0 = 0.008$, $l_0 = 0.31$, $k_0 = 0.30$. The initial condition is $y_0 = 0.1$. The CA curve is an average over 10 simulations.

In Figure 6 we display the solution of (66) with $y_0 = 0.1$ and the values of $y_k = \frac{Y_k}{N}$, where Y_k is the total number of cells occupied by cheaters at the k th-step, of a simulation on a CA with $N_L = 1599$.

5.2 Introducing fiscal policy

For simplicity we assume that the total rate of fiscal revenue of the society is proportional to the number of tax payers. Moreover, it is assumed that the total rate of expenditure is proportional to the actual value of W . Thus, the time evolution of the total wealth is given by:

$$\dot{W} = a(N - Y) - \theta W \quad (68)$$

where a and θ are non negative. Correspondingly, for the CA model, we define a normalized wealth at each time step k :

$$w_k = w_{k-1} + 1 - y_{k-1} - \theta w_{k-1} \quad (69)$$

Next we assume that the coefficients τ , α , k and l may depend on the policy the society is adopting to contrast tax evasion. More specifically, we assume that the policy of the society in controlling tax evasion is characterized in terms of the fraction $\theta \varphi$ of the budget devoted to this goal per unit time.

(i) It is reasonable to assume that α and k are increasing functions of the amount of resources the society allocates to fight tax evasion. Therefore, we assume

$$\alpha = \alpha_0 (2 - e^{-p \varphi w}) \quad (70)$$

and

$$k = k_0 (2 - e^{-p \varphi w}) \quad (71)$$

where φw accounts for the expenses devoted to prevent tax evasion and $p > 0$ is a sensitivity parameter to be adjusted to experimental data.

(ii) Coefficient τ and l on the contrary, should decrease with the intensity of the contrast to parasitism. We set

$$\tau = \tau_0 (1 + e^{-p \varphi w}) \quad (72)$$

and

$$l = l_0 (1 + e^{-p \varphi w}). \quad (73)$$

In Figure 7 we show the dependence on φ of the asymptotic values of w and of y .

One could ask which is the optimal value of φ according to this model. We give here a simple criterion: we prescribe a time horizon T (i.e. a number ν of time steps) over which we want to design our policy and give as the initial values the values \bar{y}_0, \bar{w}_0 corresponding to the “anarchic” state of the society (i.e. $\varphi = 0$). Then, we compute the time evolution of y and w for increasing values of φ . Denoting by $\bar{y}(\varphi)$ and $\bar{w}(\varphi)$ the values at step ν , the “gain” will

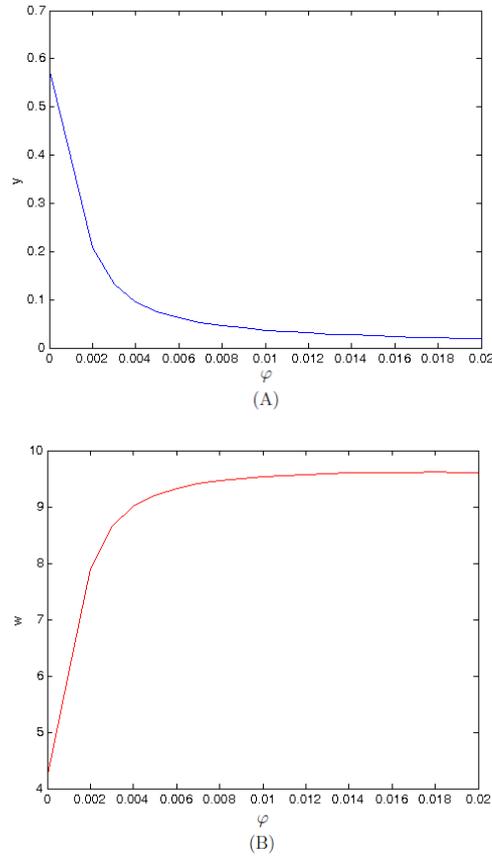


Figure 7: Asymptotic values of y (A) and w (B) obtained from the CA model for different values of φ . The parameter setup is $\alpha_0 = 0.01$ $\tau_0 = 0.008$ $l_0 = 0.31$ $m_0 = 0.30$ and $p = 5$ and $N_L = 120$. The asymptotic value is evaluated as an average over 1000 time steps (after a transient period of 1000 steps) and over 10 simulations.

be $G(\varphi) = \bar{w}(\varphi) - \bar{w}_0$. To obtain this gain, the society has spent an amount $E(\phi) = \sum_{j=1}^{n-1} \phi w_j$. In Figure 8 we display the difference $G(\varphi) - E(\varphi)$ for increasing values of φ . The simulation shows that there is a value φ_1 of φ for which this difference is maximum and a value, φ_2 , beyond which increasing investment in fighting cheaters is no longer convenient (since the difference becomes negative).

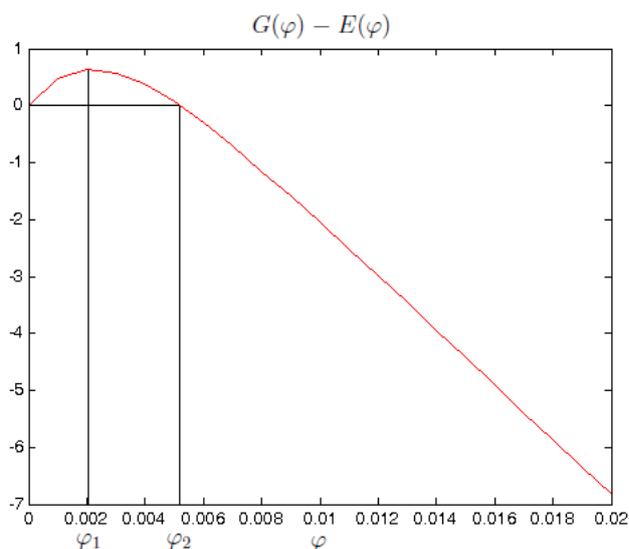


Figure 8: Difference between the gain and the expenditure over a time horizon of five years ($T = 50$). As it can be seen, $\varphi_1 \approx 0.002$ and $\varphi_2 \approx 0.005$. The same parameter setup as before is used: $\alpha_0 = 0.01$ $\tau_0 = 0.008$ $l_0 = 0.31$ $m_0 = 0.30$. Moreover, $p = 5$ and $N_l = 120$

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